J. Fluid Mech. (1993), *vol.* 255, *pp.* 91–102

Nonlinear bow flows with spray

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(Received 16 July 1992 and in revised form 24 March 1993)

The steady flow past the bow of a two-dimensional ship in water of infinite depth is considered. The ship is assumed to be a semi-infinite flat-bottomed body terminated by a face inclined at an angle β with the horizontal. The spray is modelled by a layer of water rising along the bow and falling back as a jet. A series truncation method is used to solve the fully nonlinear problem numerically. It is shown that for a prescribed value of β, there is a one-parameter family of solutions. Values of the drag and of the jet thickness are presented for different values of β.

1. Introduction

The steady, inviscid, irrotational flow past the bow of a two-dimensional ship moving at a constant velocity at the surface of a fluid of infinite depth is considered. The ship is assumed to be a semi-infinite flat-bottomed body terminated by a face inclined at an angle β with the horizontal.

Several configurations have been proposed for such flows. The simplest one is shown in figure 1(*a*): the free surface rises smoothly to a stagnation point on the bow. This flow was first proposed by Dagan & Tulin (1972), who constructed it for small Froude numbers by a perturbation expansion. It was later investigated by Vanden-Broeck & Tuck (1977), Vanden-Broeck, Schwartz & Tuck (1978) and Vanden-Broeck (1985). These authors found that waves are always present on the free surface (see figure 1(*b*)). Therefore, their solutions are appropriate for stern flows but not for bow flows. Their results show that solutions without waves (like in figure 1 *a*) are not possible in water of infinite depth. However, such flows exist in water of finite depth (Vanden-Broeck 1989).

Another configuration, also first proposed by Dagan & Tulin (1972), includes a model of the spray as a jet rising along the bow and falling down onto the oncoming stream (see figure 2). The upper free surface emanates from a stagnation point on the bow, whose position is unknown *a priori*. The impact of the jet on the oncoming flow is neglected by assuming that the jet falls into another Riemann sheet. Dagan & Tulin used matched asymptotics to construct solutions with a high Froude number. A similar analysis was used by Wu (1967), Ting & Keller (1974) and others to study the planing of flat surfaces. Fernandez (1981) extended Dagan & Tulin's calculations to three-dimensional bows. A qualitative description of the bow flows of figure 2 can also be found in Tuck & Vanden-Broeck (1985). The flow of figure 2 is not the only possible model for a bow flow. Grosenbaugh & Yeung (1989) and Yeung (1991) have studied flows by using a time-dependent scheme. Tuck & Vanden-Broeck (1985) have suggested a model in which the bow wave is approximated by a region of high vorticity, lying above an essentially irrotational flow.

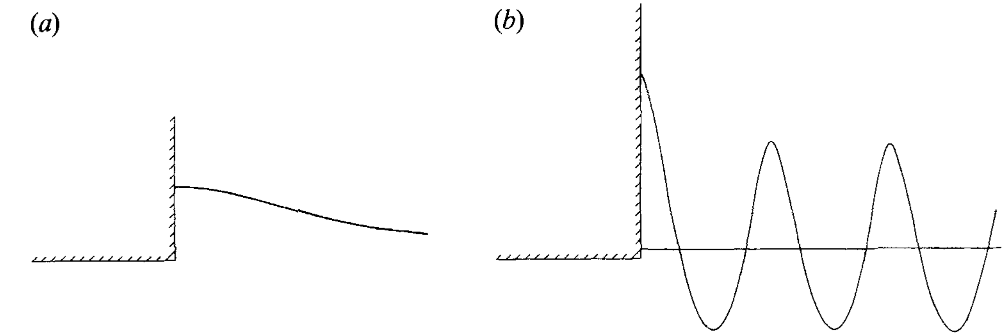

FIGURE 1. (a) Sketch of a bow flow with the free surface rising smoothly to a stagnation point on the bow. (b) Sketch of a bow flow with waves.

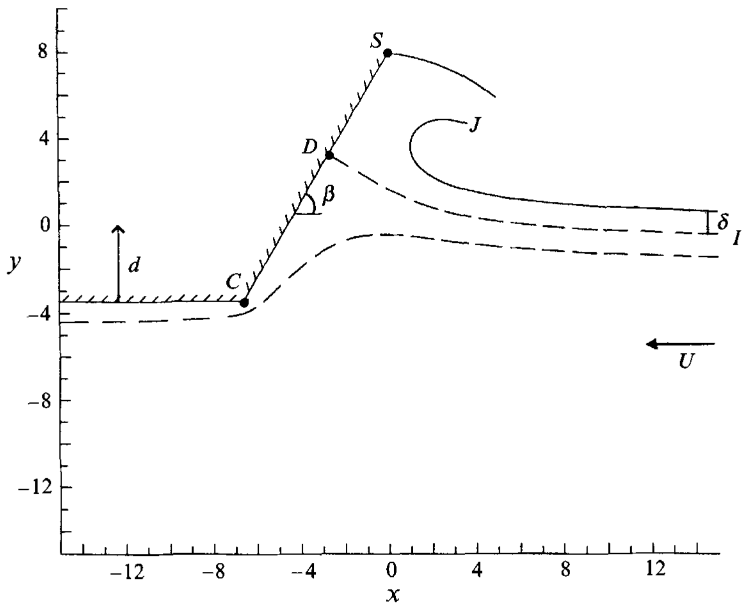

FIGURE 2. Sketch of a bow flow with spray. This is a computed solution for $\beta = \frac{1}{3}\pi$ and $F_a = 2.14$. Five points have been selected on the boundaries. The dashed lines represent the dividing streamline $\psi = 0$, which terminates on the bow, and the streamline $\psi = 1$.

In this paper, we compute solutions for the flow shown in figure 2. We assume in our formulation that there is a jet rising along the bow to a stagnation point and falling down onto the oncoming stream. The validity of this assumption is demonstrated by providing accurate numerical solutions. The numerical scheme uses series truncation and is similar to the procedure used by Vanden-Broeck & Keller (1987), Dias, Keller & Vanden-Broeck (1988) and Dias & Tuck (1991) to investigate weir flows and waterfalls. Dias & Christodoulides (1991) and Dias & Tuck (1993) have adapted these schemes to compute spray-like flows in a corner, where the whole oncoming flow rises along the wall. They were able to model the loop that the sheet of water makes as it falls back onto the oncoming flow, and found that such flows exist only above a critical value of the Froude number. Vanden-Broeck (1993) computed a similar flow using finite differences.

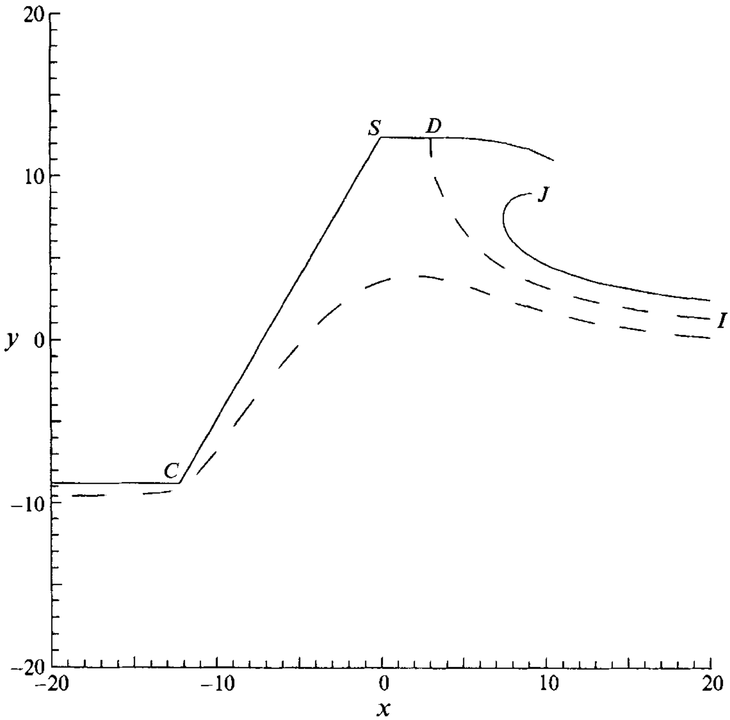

FIGURE 3. As figure 2 but for $F_d = 1.68$. The dashed lines represent the dividing streamline $\psi = 0$, which ends on the upper free surface, and the streamline $\psi = 1$.

It is convenient to describe the flow in terms of the angle β between the bow and the horizontal, and of the two Froude numbers

$$F_d = U/(gd)^{\frac{1}{2}}, \tag{1.1}$$

and

$$F_\delta = U/(g\delta)^{\frac{1}{2}}, \tag{1.2}$$

where U is the velocity of the ship, g the acceleration due to gravity, d the draught and δ the upstream thickness of the jet rising along the bow. Now that the ratio between the jet thickness and the draught is

$$\frac{\delta}{d} = \left(\frac{F_d}{F_\delta}\right)^2. \tag{1.3}$$

As we shall see, there is a one-parameter family of solutions for each given value of β. The parameter can be chosen as the draught Froude number (1.1). Results will be presented for different values of β. As F_d decreases, the stagnation point D at the end of the dividing streamline first rises along the bow and then moves along the upper free surface (see figure 3). When D is on the upper free surface, there are two stagnation points on the upper free surface and there is therefore a region in between where the fluid is almost stagnant.

The problem is formulated in §2. Use is made of the hodograph variable. In §3 the problem is solved numerically by a series truncation method. In §4 the results are discussed and plots of various physical quantities are shown. In §5, an exact solution to the problem in the absence of gravity is given.

2. Formulation of the problem

The steady irrotational flow of an incompressible inviscid fluid past the bow of a ship is considered (see figure 2). The ship is assumed to be semi-infinite. The bow makes an angle β with the horizontal. In a frame of reference moving with the ship, we study the problem of a uniform flow approaching the ship from the right with velocity $-U$. The water is supposed to be of infinite depth. We denote infinity upstream by I, the stagnation point along the bow by S, the stagnation point where the dividing streamline ends by D and the intersection between the bow and the bottom of the ship by C. We will first formulate the problem by assuming that the stagnation point D lies on the bow. It will be shown later that D can also lie on the upper free surface. The corresponding changes in the formulation will be described in the next section. As the flow approaches the bow, part of it rises along the bow and eventually comes to rest. It then becomes a jet which falls down to infinity (point J). It is assumed here that the jet does not cross the oncoming flow. The problem is non-dimensionalized by taking U as the unit velocity and the upstream thickness δ of the jet as the unit length. In dimensionless coordinates, Bernoulli's equation on the free surfaces takes the form

$$q^2+\frac{2y}{F_\delta^2}=1, \tag{2.1}$$

if we choose the horizontal x-axis to be along the lower free surface at infinity. The vertical y-axis goes through the point S and q is the magnitude of the velocity.

We denote the velocity potential by $\phi(x,y)$ and the stream function by $\psi(x,y)$. In addition we introduce the complex variables $z=x+\mathrm{i}y$ and $f=\phi+\mathrm{i}\psi$. The flow domain in the f-plane is shown in figure 4(*a*). There is a slit starting at D. Because of the choice of dimensionless variables, the distance between the streamline IJ and the slit (i.e. the flux going into the jet) is 1.

Following the method that was used successfully in other papers quoted in the introduction, the domain occupied by the fluid in the f-plane is mapped onto the upper half of the unit disk in an auxiliary t-plane so that the points S, I and J are mapped into the points -1, 1 and i (see figure 4*b*). The bottom and the bow of the ship go onto the real diameter. The images of the points C and D are denoted by t_c and t_d. The transformation from the f-plane to the t-plane can be written in differential form as

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{4}{\pi(1+t_d^2)}\left[\frac{(t-t_d)(1-tt_d)(1+t)}{(1-t)^3(1+t^2)}\right]. \tag{2.2}$$

The hodograph variable

$$\zeta(z)\overset{\mathrm{def}}{=}\frac{\mathrm{d}f}{\mathrm{d}z}(z) \tag{2.3}$$

is then introduced. The problem to be solved is to find ζ as a function of t which is analytic inside the upper half-unit disk and continuous on the boundaries. Moreover, Bernoulli's equation must be satisfied on the free surfaces and kinematic boundary conditions must be satisfied on the real diameter. Points on the free surfaces are represented by $t=\mathrm{e}^{\mathrm{i}\sigma}$.

The complex function ζ is singular at the points S, D, C and J. At point S $(t=-1)$, where the bow intersects the upper free surface, the velocity vanishes. The local behaviour of the flow near S has been studied by Dagan & Tulin (1972). Their analysis implies that the appropriate singularity is

$$\zeta\sim(t+1)^{2\tau/\pi}\quad\text{as}\quad t\to-1,$$

where $\tau=\beta$ if $\frac{1}{3}\pi\leqslant\beta\leqslant\frac{1}{2}\pi$ and $\tau=\frac{1}{3}\pi$ if $0<\beta\leqslant\frac{1}{3}\pi$.

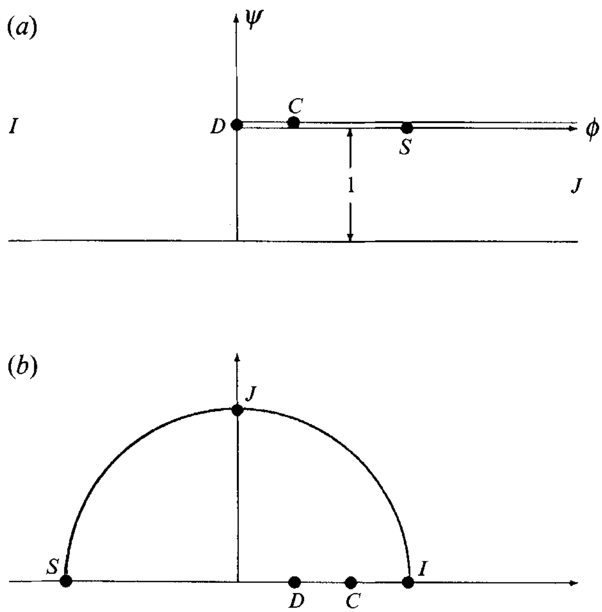

FIGURE 4. Flow domain in: (*a*) the plane of the complex potential f, (*b*) the intermediate t-plane. This case corresponds to bow flows where the dividing streamline ends along the bow. The images of the five selected points of figure 2 are shown.

The free surface is horizontal at S when $\frac{1}{3}\pi \leqslant \beta \leqslant \frac{1}{2}\pi$ and forms an angle of $\frac{2}{3}\pi$ with the bow when $0 < \beta \leqslant \frac{1}{3}\pi$.

At point D, where the dividing streamline intersects the bow, the velocity vanishes and

$$\zeta \sim (t-t_d) \quad \text{as} \quad t \to t_d.$$

At the corner C between the bow and the bottom of the ship, the velocity is infinite. Since there is a corner with an angle β, the appropriate singularity is

$$\zeta \sim (t-t_c)^{-\beta/\pi} \quad \text{as} \quad t \to t_c.$$

At point J ($t = \mathrm{i}$), there is a jet-type singularity. It can be shown (see Birkhoff & Carter 1957 and Vanden-Broeck & Keller 1986 for details) that the behaviour of ζ is

$$\zeta \sim [\ln(t^2+1)]^{\frac{1}{3}} \quad \text{as} \quad t \to \mathrm{i}.$$

Taking into account the above singularities, the hodograph variable is written as

$$\zeta = -\left(\frac{t+1}{2}\right)^{2\tau/\pi} \frac{(t-t_d)(1-t_d t)}{(1-t_d)^2} \left(\frac{t-t_c}{1-t_c}\right)^{-\beta/\pi} \left[\frac{\ln c(1+t^2)}{\ln 2c}\right]^{\frac{1}{3}} \left[1 + \sum_{n=1}^{\infty} a_n(t^n - 1)\right]. \quad (2.4)$$

With such an expression, the velocity is automatically -1 at infinity (i.e. at $t = 1$). In (2.4), c is an arbitrary constant smaller than 0.5. We checked that the final numerical results are independent of the value of c. In most of the calculations, we chose $c = 0.2$.

The coefficients a_n in (2.4) are to be found to satisfy (2.1) on the free surfaces.

3. Numerical solution

We solve the problem numerically by truncating the infinite series in (2.4) after a finite number of terms. As we shall see, there is, for each value of β, a one-parameter family of solutions. It is convenient to choose this parameter as the image t_d of the stagnation point D in the t-plane. For a given value of t_d, F_δ and t_c are determined as part of the solution. We truncate the infinite series in (2.4) after $(N-2)$ terms, so that there are N unknowns: F_δ, t_c and the $(N-2)$ coefficients a_n. We introduce $(N-1)$ mesh points on the free surfaces, $\frac{1}{2}(N-1)$ on each (assuming that N is odd). These points are defined by

$$\sigma_m = \frac{\pi}{N-1}(m-\tfrac{1}{2}), \tag{3.1}$$

with $1 \leqslant m \leqslant \frac{1}{2}(N-1)$ for the lower free surface and $\frac{1}{2}(N+1) \leqslant m \leqslant (N-1)$ for the upper free surface. We want to satisfy (2.1) at each of the collocation points. Therefore, we need to evaluate the elevation y at the mesh points (3.1). This is done as follows. For points on the upper free surface, we use (2.3) to write

$$x(\sigma_m)+\mathrm{i}y(\sigma_m) = \mathrm{i}\frac{F_\delta^2}{2} - \int_{\sigma_m}^{\pi} \frac{1}{\zeta}\frac{\mathrm{d}f}{\mathrm{d}\sigma}\,\mathrm{d}\sigma. \tag{3.2}$$

The integration starts at $\sigma = \pi$ and proceeds with the trapezoidal rule. For points on the lower free surface, we obtain z as

$$x(\sigma_m)+\mathrm{i}y(\sigma_m) = x(\sigma^*)+\mathrm{i}y(\sigma^*) + \int_{\sigma^*}^{\sigma_m} \frac{1}{\zeta}\frac{\mathrm{d}f}{\mathrm{d}\sigma}\,\mathrm{d}\sigma, \tag{3.3}$$

where σ^* is an arbitrary value of σ_m between 0 and $\frac{1}{2}\pi$. The value of $z(\sigma^*)$ is obtained by integrating directly across the unit disk from $t = -1$ to $t^* = \mathrm{e}^{\mathrm{i}\sigma^*}$:

$$z(\sigma^*) = \mathrm{i}\frac{F_\delta^2}{2} - \int_{t^*}^{\pi} \frac{1}{\zeta}\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t. \tag{3.4}$$

Bernoulli's equation (2.1) can now be satisfied at the mesh points (3.1). This yields $(N-1)$ equations. The last equation is obtained by imposing

$$\left.\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right|_{t=1} = 0, \tag{3.5}$$

so that the free surface is flat upstream, i.e. without waves.

For given values of β and t_d, the system of N nonlinear equations is solved by Newton's method, using the package C05NBF from the NAG library. Once the system is solved, the coordinates of the points C and D are computed by integrating $\mathrm{d}z/\mathrm{d}f$ along the bow. Next, the jet thickness (1.3) and the draught Froude number (1.1) are calculated as follows:

$$\delta/d = -1/y(C), \tag{3.6}$$

$$F_d = F_\delta/(-y(C))^{\frac{1}{2}}. \tag{3.7}$$

Finally, the drag Δ exerted on the bow is evaluated. It is equal to the horizontal component of the pressure force exerted on the bow. Using Bernoulli's equation, one obtains the following expression for the drag coefficient:

$$\frac{\Delta}{\rho U^2 \delta} = \frac{1}{2}\int_{y(C)}^{y(S)} \left(1-q^2-\frac{2y}{F_\delta^2}\right)\mathrm{d}y. \tag{3.8}$$

4. Discussion of the results

We use the scheme described in §3 to compute solutions for various values of t_d and β. Profiles of the free surfaces, the dividing streamline and the streamline $\psi = 1$ are shown in figures 2, 3 and 5. Table 1 shows the accuracy of the results as a function of N for $\beta = \frac{1}{2}\pi$ and $t_d = 0.7$. Solutions were computed for values of t_d between -1 and $+1$ and for various values of β less than $\frac{1}{2}\pi$.

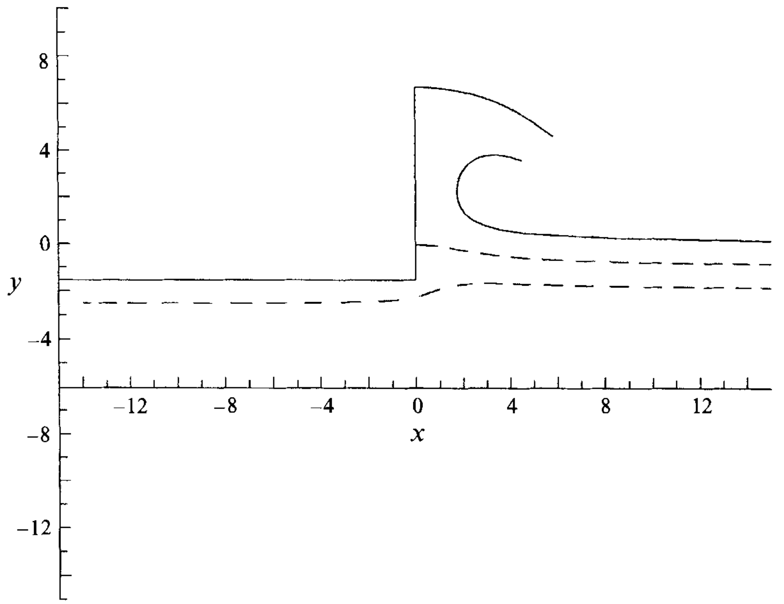

FIGURE 5. Computed solution with $t_d = 0.935$. The resulting values are $t_c = 0.971$, $F_\delta = 3.66$, $F_d = 2.95$. The dividing streamline $\psi = 0$ and the streamline $\psi = 1$ are shown.

N	F_δ	t_c	$y(C)$	$y(D)$	F_d
101	3.59	0.956	−1.56	0.02	2.88
161	3.59	0.956	−1.57	0.02	2.86
301	3.58	0.956	−1.60	−0.01	2.83
401	3.58	0.956	−1.59	−0.01	2.84

TABLE 1. Accuracy of computed quantities as a function of N. The given value of t_d is 0.7 and $\beta = \frac{1}{2}\pi$. The values of the coefficients are $a_1 = 0.38$, $a_{10} \sim 0.005$, $a_{100} \sim 0.0001$.

As t_d approaches $+1$, both Froude numbers F_d and F_δ become very large. The vertical coordinate of $y(C)$ approaches -1, i.e. the draught and the upstream thickness of the jet become equal. The limiting configuration for $t_d = +1$ is the no-gravity solution described in §5.

The draught Froude number F_d decreases as t_d decreases. As t_d approaches -1, the stagnation point D moves closer to the stagnation point S, and the two stagnation points merge into one for $t_d = -1$. Even so, the program still converged for $t_d = -1$! (see figure 6).

The family of solutions can be continued if D is allowed to lie on the upper free surface. The changes involved in the formulation are minor. The image of the point D in the t-plane now lies on the unit circle and can be expressed as $t_d = e^{i\gamma}$, with γ between $\frac{1}{2}\pi$ and π. The corresponding f- and t-planes are shown in figure 7. The expression (2.2) for df/dt becomes

$$\frac{df}{dt} = -\frac{2}{\pi\cos\gamma}\left[\frac{(1+t^2-2t\cos\gamma)(1+t)}{(1-t)^3(1+t^2)}\right], \tag{4.1}$$

while the expression (2.4) for ζ becomes

$$\zeta = -\left(\frac{t+1}{2}\right)^{2\gamma/\pi}\left(\frac{1+t^2-2t\cos\gamma}{2(1-\cos\gamma)}\right)\left(\frac{t-t_c}{1-t_c}\right)^{-\beta/\pi}\left[\frac{\ln c(1+t^2)}{\ln 2c}\right]^{\frac{1}{3}}\left[1+\sum_{n=1}^{\infty} a_n(t^n-1)\right]. \tag{4.2}$$

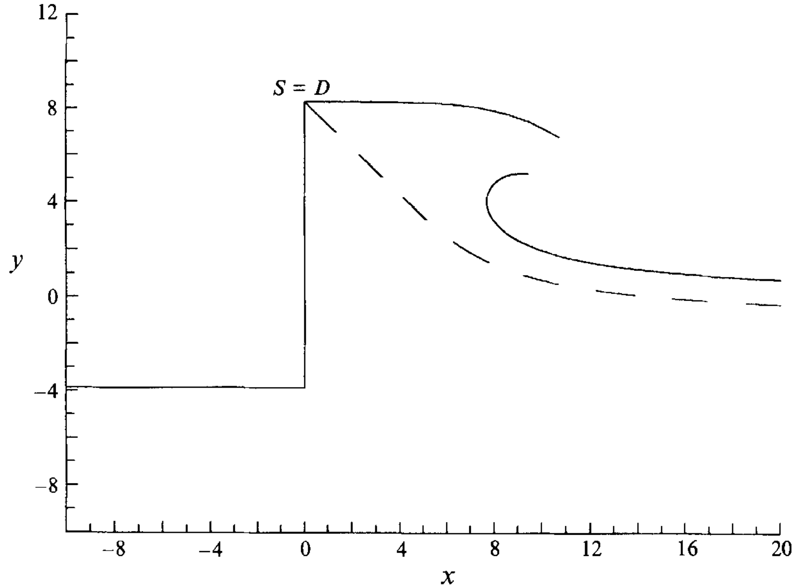

FIGURE 6. Computed solution with $t_a = -1$. The resulting values are $t_c = 0.587$, $F_\delta = 4.07$, $F_a = 2.07$. The dividing streamline $\psi = 0$ is shown.

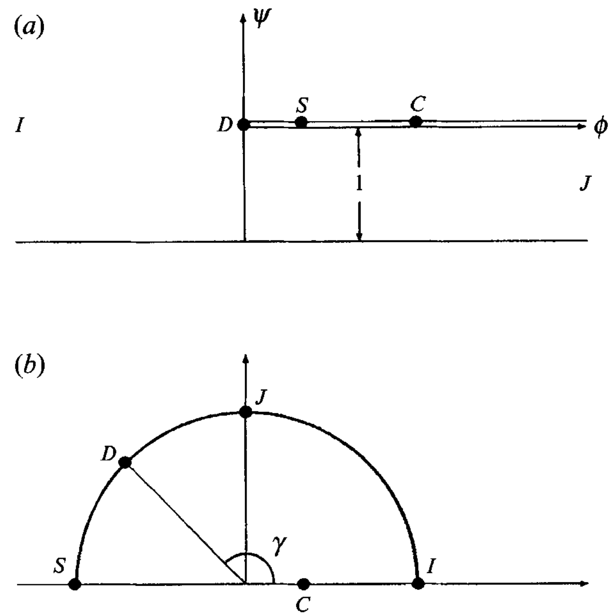

FIGURE 7. Flow domain in: (a) the plane of the complex potential f, (b) the intermediate t-plane. This case corresponds to bow flows where the dividing streamline ends on the upper free surface. The images of the five selected points of figure 3 are shown.

A computed solution for $\gamma = \frac{2}{3}\pi$ is shown in figure 8. The free surface goes down very slightly between the two stagnation points S and D but it cannot be seen on the figure. Accurate solutions can be calculated for γ greater than $\approx 0.6\pi$. For γ smaller than $\approx 0.6\pi$, the convergence of the scheme as N increases becomes poor. The reason is that

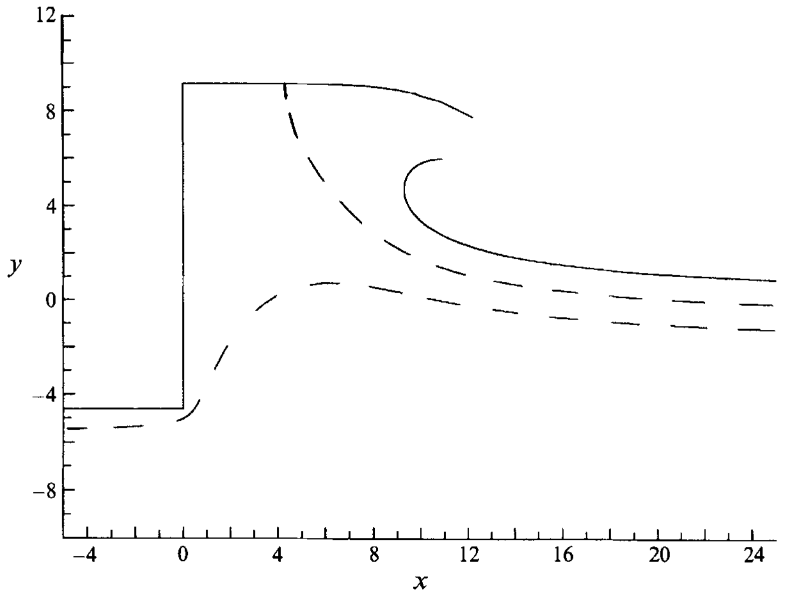

FIGURE 8. Computed solution with $\gamma = \frac{2}{3}\pi$. The resulting values are $t_c = 0.548$, $F_\delta = 4.28$, $F_d = 1.99$. The dividing streamline $\psi = 0$ and the streamline $\psi = 1$ are shown.

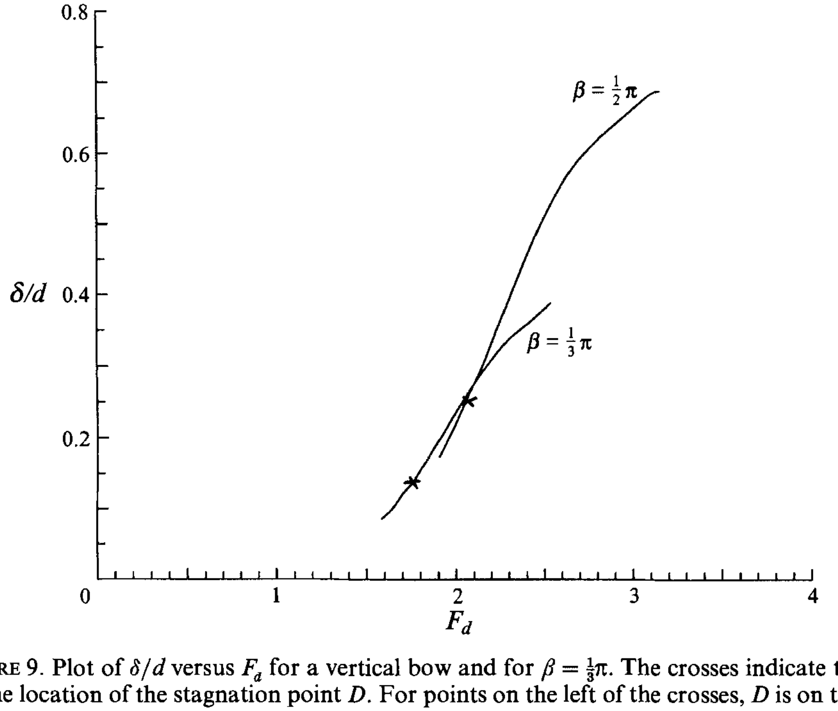

FIGURE 9. Plot of δ/d versus F_d for a vertical bow and for $\beta = \frac{1}{3}\pi$. The crosses indicate the transition for the location of the stagnation point D. For points on the left of the crosses, D is on the upper free surface. For points on the right of the crosses, D is on the bow.

there are not enough mesh points between D and J. Investigation of the solutions for $\gamma < 0.6\pi$ is left for future work.

Figure 9 shows a plot of δ/d versus F_d for a vertical bow and for $\beta = \frac{1}{3}\pi$. As the draught Froude number decreases, the jet thickness decreases. As $F_d \to \infty$, the limit

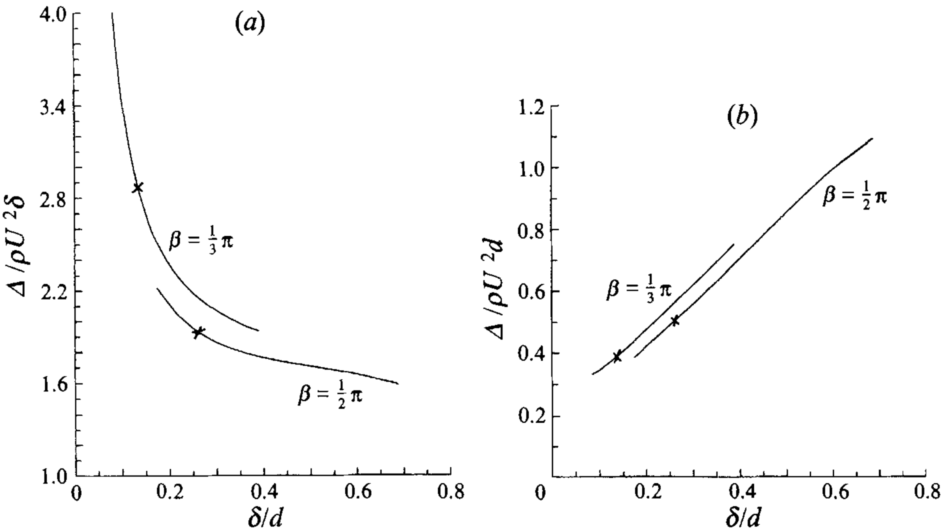

FIGURE 10. (a) Plot of the drag coefficient (see (3.8)) versus δ/d for a vertical bow and $\beta = \frac{1}{3}\pi$; (b) plot of the drag coefficient defined as $\Delta/\rho U^2 d$. The crosses indicate the transition for the location of the stagnation point D. For points on the left of the crosses, D is on the upper free surface. For points on the right of the crosses, D is on the bow.

is $\delta/d = 1$. A plot of the drag coefficient (3.8) versus the jet thickness for the same values of β is shown in figure 10(a). The drag coefficient approaches the value $1+\cos\beta$ as the jet thickness approaches one. A plot of the drag coefficient defined as $\Delta/\rho U^2 d$ is shown in figure 10(b).

5. Solution without gravity

As t_a approaches 1, the flow approaches the solution without gravity shown in figure 11. The flow rises indefinitely along the bow without falling. There is only one free surface along which the magnitude of the velocity is equal to 1. In order to calculate this flow, the f-plane and the t-plane must be slightly modified as shown in figure 12. As was shown by Oertel (1975) and Dias & Elcrat (1992), there is an exact solution for ζ, namely

$$\zeta = -t\left(\frac{t-t_c}{1-t_c t}\right)^{-\beta/\pi}. \tag{5.1}$$

Since

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{16}{\pi}\frac{t}{(1-t)^3(1+t)}, \tag{5.2}$$

it follows that

$$z = -\frac{16}{\pi}\int_{t_c}^{t}\frac{1}{(1-t)^3(1+t)}\left(\frac{t-t_c}{1-t_c t}\right)^{\beta/\pi}\mathrm{d}t. \tag{5.3}$$

There are two parameters in the problem: the angle β and t_c. However, the number of parameters can be reduced to one if one requires the free surface to become horizontal upstream. This requirement is satisfied if the derivative $\mathrm{d}\zeta/\mathrm{d}t$ vanishes at $t = 1$. It is easy to show that it is the case if

$$t_c = \frac{\pi-\beta}{\pi+\beta}. \tag{5.4}$$

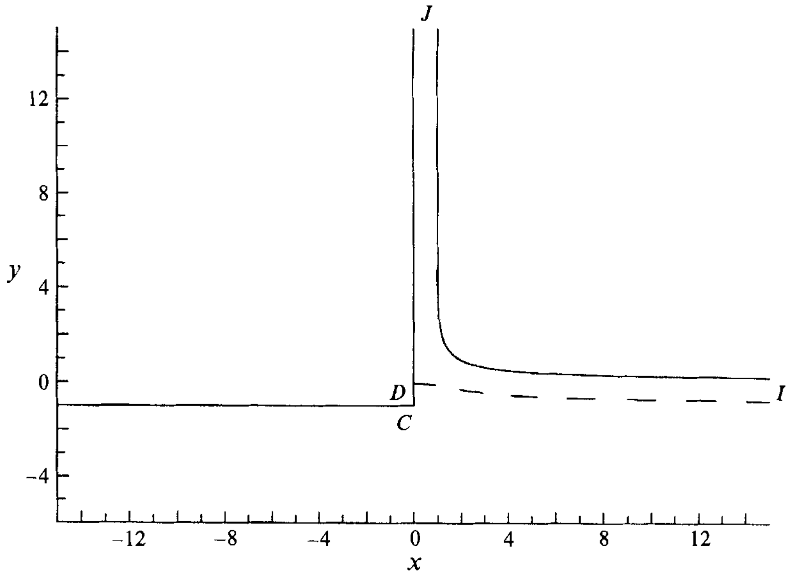

FIGURE 11. Computed solution without gravity for a vertical wall. The dashed line is the dividing streamline.

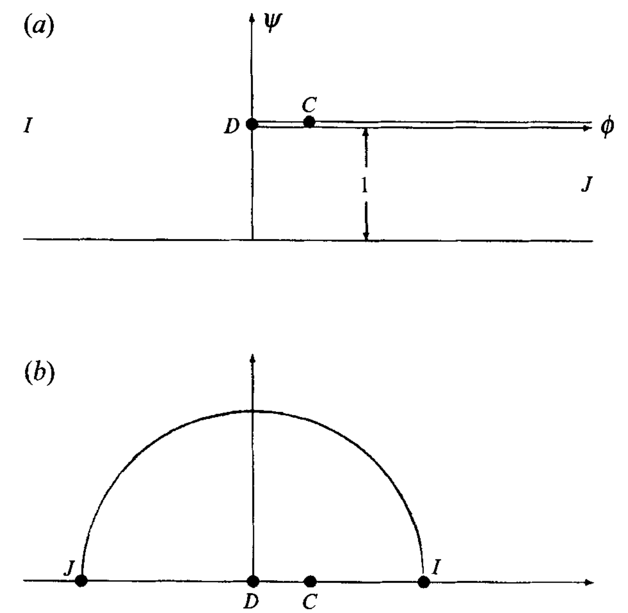

FIGURE 12. Flow domain in: (*a*) the plane of the complex potential f, (*b*) the intermediate t-plane. This case corresponds to flows without gravity. The images of the four selected points of figure 11 are shown.

The horizontal component of the pressure force Δ acting on the boundary CDJ is easily found from a horizontal momentum balance. One finds

$$\frac{\Delta}{\rho U^2 \delta} = \frac{1}{2}\int_{y(C)}^{\infty} (1-q^2)\,\mathrm{d}y = 1+\cos\beta. \tag{5.5}$$

A plot of the free surface and of the dividing streamline is shown in figure 11 for $(\beta, t_c) = (\frac{1}{2}\pi, \frac{1}{3})$. The coordinates of the stagnation point terminating the dividing streamline are (0, -0.0463). The ratio δ/d is equal to 1 exactly.

The authors are grateful for a number of helpful comments from E. O. Tuck. In particular, he suggested that the Froude number F_d corresponding to $t_d = -1$ was not a lower bound, but that solutions with smaller Froude numbers were likely to exist. These solutions turned out to be the solutions with the dividing streamline terminating on the free surface and not on the bow.

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